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Substituting into (6′) we have

$$\left[Pf\left(x-\frac{Pf}{p}a\right)\right]_0+\left[P\left(x-\frac{2Pf}{p}a\right)^2-\frac{P^2}{p}f^2\right]_0\varDelta a_0 + \left[P\left(x-\frac{2Pf}{p}a\right)\right]_0\varDelta b_0=0, \quad (m_a)$$

$$\left[Pf\right]_0+\left[P\left(x-\frac{2Pf}{p}a\right)\right]_0\varDelta a_0+\left[P\right]_0\varDelta b_0=0. \quad (m_b)$$

Employing (9) (6″) and (10) we have also

$$X_1=x_1-\frac{P_1f_1}{p_1}a\,;\ Y_1=1, \qquad (n_1)$$

$$X_2=x_2-\frac{P_2f_2}{p_2}a\,;\ Y_2=1, \qquad (n_2)$$

. . . . . .

Should the results of these two methods differ materially, the indication is that there is considerable uncertainty about the determination of the unknown quantities or that their probable errors are large. It would not be judicious in that case to attempt a more exact solution.

SYMMETRICAL FUNCTIONS OF THE SINES OF THE ANGLES INCLUDED IN THE EXPRESSION

$$\alpha_0+\frac{2k\pi}{n}.$$

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DENOTING these angles by $\alpha_0, \alpha_1, \alpha_2, \ldots \alpha_{n-1}$, the values of $\sin n\alpha$ and $\cos n\alpha$ will be the same for all; hence, putting x for $\sin\theta$ and $n\alpha$ for $n\theta$ in the developments of $\cos n\theta$ and $\sin n\theta$ in terms of $\sin\theta$ (quoted on page 17) we have the eq'ns whose roots are $\sin\alpha_0$, $\sin\alpha_1$, etc.; viz., when n is even,

$$\cos n\alpha=1-\frac{n^2}{2}x^2+\frac{n^2(n^2-2^2)}{4!}x^4\ldots+(-1)^{\frac{1}{2}n}\frac{n^2(n^2-2^2)(n^2-4^2)\ldots[n^2-(n-2)^2]}{n!}x^n, \tag{1}$$

and when n is odd,

$$\sin n\alpha=nx \qquad -\frac{n(n^2-1^2)}{3!}x^3\ldots+(-1)^{\frac{n-1}{2}}\frac{n(n^2-1^2)(n^2-3^2)\ldots[n^2-(n-2)^2]}{n!}x^n. \tag{2}$$

On dividing by the coefficient of x^n each of these eq'ns takes the form

$$0=x^n-\frac{n(n-1)}{n^2-(n-2)^2}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{[n^2-(n-2)^2][n^2-(n-4)^2]}x^{n-4}-\ldots,$$

the equations differing only in the last terms. Reducing the coefficients, the equation becomes

$$0 = x^n - \frac{n}{4}x^{n-2} + \frac{n(n-3)}{4.8}x^{n-4} - \frac{n(n-4)(n-5)}{4.8.12}x^{n-6}, \ldots \tag{3}$$

the general term being

$$(-1)^r \frac{n(n-r-1)\ldots(n-2r+1)}{4^r.r!}x^{n-2r}.$$

But when n is even the last term, [as found from (1)] is obtained by putting $1-\cos na$ in place of x^0 when $r = \frac{1}{2}n$; it is therefore

$$(-1)^{\frac{1}{2}n} \frac{n(\frac{1}{2}n-1)!}{2^n.(\frac{1}{2}n)!}(1-\cos na) = (-1)^{\frac{1}{2}n} \frac{\sin^2 \frac{1}{2}na}{2^{n-2}}: \tag{4}$$

and when n is odd the last term [as found from (2)] is obtained by putting $-\sin na$ in place of nx in the term corresponding to $r = \frac{1}{2}(n-1)$; the last two terms are therefore

$$(-1)^{\frac{n-1}{2}} \frac{n.\left(\frac{n-1}{2}\right)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}x - (-1)^{\frac{n-1}{2}} \frac{\sin na}{2^{n-1}}. \tag{5}$$

Since the last terms only involve a, all the symmetrical rational integral functions of $\sin a_0$, $\sin a_1$, etc., *of a degree less than n* have values independent of a_0, and the form of equation (3) shows that all those of an odd order vanish. With this restriction we have

$$\Sigma \sin a_0 \sin a_1 = \tfrac{1}{4}n, \quad \Sigma \sin a_0 \sin a_1 \sin a_2 \sin a_3 = \tfrac{1}{32}n(n-3),$$

and in general

$$\Sigma \sin a_0 \ldots\ldots \sin a_{2r-1} = (-1)^r \frac{n.(n-r-1)!}{4^r.r!(n-2r)!}.$$

By employing Hirsch's tables of symmetrical functions (given in an Appendix to Salmon's Higher Algebra) we may calculate any function of degree not higher than the tenth, and any function involving only even powers, up to the twentieth degree. I had by this means obtained

$$\Sigma \sin^2 a = \tfrac{1}{2}n, \quad \Sigma \sin^4 a = \tfrac{3}{8}n, \quad \Sigma \sin^6 a = \tfrac{5}{16}n, \quad \Sigma \sin^8 a = \tfrac{35}{128}n,$$

which are remarkable as containing the first power only of n, before I noticed that it is easily shown *a priori* that this is universally true. For compare the values of $\Sigma \sin^{2r} a$ for two values of n, n' and n'', (both greater than $2r$) with its value when $n = n'n''$. It is evident that the angles $a_0 + (2k\pi \div n'n'')$ are equivalent to n'' sets of angles of the form $a'_0 + (2k\pi \div n')$, hence the $n'n''$ terms of $\Sigma^{n'n''}\sin^{2r} a$ are equivalent to n'' sets of terms each of which has for its sum $\Sigma^{n'}\sin^{2r} a$; that is, $\Sigma^{n'n''}\sin^{2r} a = n''\Sigma^{n'}\sin^{2r} a = n'\Sigma^{n''}\sin^{2r} a$. Therefore $\Sigma^{n'}\sin^{2r} a = kn'$, where k is a numerical constant.

The value of $\Sigma \sin^{2r}a$ is now easily calculated since it is necessary to retain only the first power of n in the result. Since each of the coefficients in eq. (3) contains n as a factor, it is unnecessary to retain any terms in the expression for $\Sigma \sin^{2r}a$ in terms of the coefficients which involve more than one coefficient. Now the only term of Σa^{2r} which is of the first degree in the coefficients is $-2rp_{2r}$ (p_{2r} being the coefficient of x^{n-2r} in the equation whose roots are a, β, etc). Moreover, in the value of p_2(see general term above) it is only necessary to retain the numerical part of each of the binomial factors. Therefore

$$\Sigma \sin^{2r}a = -2rp_{2r}+\&c. = \frac{2rn.(r+1)\ldots(2r-1)}{4^r.r\,!} = \frac{n}{2^{2r-1}}\cdot\frac{(2r-1)\,!}{r\,!\,(r-1)\,!}\cdot$$

the coefficient of $n\div 2^{2r-1}$ being the middle coefficient of the expansion of $(a+b)^{2r-1}$.

We may now readily calculate any symmetrical function by means of the

formulæ $\qquad \Sigma a^m\beta^p = \Sigma a^m.\Sigma a^p - \Sigma a^{m+p}$, &c.;

for example $\qquad \Sigma\sin^2 a_0.\sin^6 a_1 = \frac{5}{32}n^2 - \frac{35}{128}n$,

and $\qquad \Sigma\sin^3 a_0.\sin^5 a_1 = -\frac{35}{128}n$.

The values (4) and (5) of the last term of equation (3) show that, when n is even, the product

$$\sin a_0.\sin a_1 \ldots \sin a_{n-1} = (-1)^{\frac{1}{2}n} \sin^2\tfrac{1}{2}na \div 2^{n-2},$$

and when n is odd

$$\sin a_0.\sin a_1 \ldots \sin a_{n-1} = (-1)^{\frac{1}{2}(n-1)} \sin na \div 2^{n-1}.$$

In the first of these expressions one half of the factors are negative, and numerically equal to the positive factors; hence putting $2n'$ in place of n, the product of the sines of the angles included in the expres'n $a_0 + (k\pi \div n')$ $[a_0 < (\pi \div n)]$, is $\sin a_0.\sin a_1 \ldots \sin a_{n'-1} = \sin n'a \div 2^{n'-1}$.

These expressions of course vanish when $a_0 = 0$, but in that case if we divide by $\sin a_0$ and evaluate, we have, when n is odd,

$$\sin\frac{2\pi}{n}\sin\frac{4\pi}{n}\ldots\sin\frac{2(n-1)\pi}{n} = (-1)^{\frac{n-1}{2}}\frac{n}{2^{n-1}},$$

and for all values of n'

$$\sin\frac{\pi}{n'}, \text{Sin}\frac{2\pi}{n'}, \ldots \sin\frac{(n'-1)\pi}{n'} = \frac{n'}{2^{n'-1}}.$$

Equation (1) shows that the sum of the reciprocals of the roots, $\Sigma \operatorname{cosec} a_0$ vanishes when n is even, but

$\Sigma \operatorname{cosec} a_0 \operatorname{cosec} a_1 = -n^2 \div (4\sin^2\frac{1}{2}na)$, whence $\Sigma \operatorname{cosec}^2 a = \frac{1}{2}n^2 \operatorname{cosec}^2\frac{1}{2}na$: and equation (2) shows that when n is odd,

$$\Sigma \operatorname{cosec} a_0 = n \operatorname{cosec} na,\quad \Sigma \operatorname{cosec} a_0 \operatorname{cosec} a_1 = 0, \text{ etc.}$$